

More Properties on Multi Poly-Euler Polynomials

Hassan Jolany and Roberto B. Corcino

Abstract

In this paper, we establish more properties of generalized poly-Euler polynomials with three parameters and we investigate a kind of symmetrized generalization of poly-Euler polynomials. Moreover, we introduce a more general form of multi poly-Euler polynomials and obtain some identities parallel to those of the generalized poly-Euler polynomials.

Mathematics Subject Classification (2010). 11B68, 11B73, 05A15.

Keywords: poly-Euler polynomials, Appell polynomials, poly-logarithm, generating function.

1 Introduction

The nature of introducing Bernoulli numbers is parallel to that of Euler numbers. Bernoulli numbers have been introduced by Jacob Bernoulli (1655-1705) in his effort to describe the coefficients of the polynomial representation of the sum

$$S_m(n) = 1^m + 2^m + \dots + n^m,$$

while Euler numbers have been introduced by Leonard Euler (1707-1783) in his desire to evaluate the alternating sum

$$A_n(m) = m^n - (m-1)^n + \dots + (-1)^{m-1}1^n.$$

Moreover, Bernoulli and Euler numbers have been usually defined by means of the following generating functions with parallel structures

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{1}$$

$$\frac{2}{e^x + 1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}, \tag{2}$$

respectively. It is worth mentioning that Euler worked intensively on Bernoulli numbers and gave great contributions in the development of the numbers. This made him known as “godfather” of Bernoulli numbers (see [4]).

In 1997, Kaneko [11] introduced the poly-Bernoulli numbers $B_n^{(k)}$ by means of the following exponential generating function

$$\frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}$$

where

$$\text{Li}_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^k}.$$

Ohno and Sasaki [16], on the other hand, defined poly-Euler numbers as

$$\frac{\text{Li}_k(1 - e^{-4t})}{4t \cosh t} = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}$$

which have been recently extended by H. Jolany et al. [15] in polynomial form as

$$\frac{2\text{Li}_k(1 - e^{-t})}{1 + e^t} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}. \quad (3)$$

Further generalization and other properties of poly-Bernoulli and poly-Euler numbers and polynomials including their relations with other special numbers and functions are found in [1, 2, 3, 5, 6, 8, 9, 10, 12].

In [15], the generalized poly-Euler polynomials with parameters a , b and c are defined by

$$\frac{2\text{Li}_k(1 - (ab)^{-t})}{a^{-t} + b^t} c^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x; a, b, c) \frac{t^n}{n!}. \quad (4)$$

Note that the poly-Euler polynomials in [2, 16] can be deduced from (4) by replacing t with $4t$ and taking $x = 1/2$. Moreover, when $x = 0$, (4) gives

$$E_n^{(k)}(0; a, b, c) = E_n^{(k)}(a, b)$$

where

$$\frac{2\text{Li}_k(1 - (ab)^{-t})}{a^{-t} + b^t} = \sum_{n=0}^{\infty} E_n^{(k)}(a, b) \frac{t^n}{n!},$$

and when $a = 1$ and $b = c = e$ with $E_n^{(k)}(x; 1, e, e) = E_n^{(k)}(x)$, we obtain equation (3). However, only one identity has been obtained in [15] for $E_n^{(k)}(x; a, b, c)$ which is given by

$$E_n^{(k)}(x; a, b, c) = \sum_{m=0}^n \sum_{j=0}^m \sum_{i=0}^j \frac{2(-1)^{m-j+i}}{j^k} \binom{j}{i} (x \ln c - (m-j+i+1) \ln a - (m-j+i+1) \ln b)^n. \quad (5)$$

In the same paper by H. Jolany et. al [15], they defined certain multi poly-Euler polynomials as follows

$$\frac{2\text{Li}_{(k_1, k_2, \dots, k_r)}(1 - (ab)^{-t})}{(a^{-t} + b^t)^r} e^{rxt} = \sum_{n=0}^{\infty} E_n^{(k_1, k_2, \dots, k_r)}(x; a, b) \frac{t^n}{n!}. \quad (6)$$

where

$$\text{Li}_{(k_1, k_2, \dots, k_r)}(z) = \sum_{0 \leq m_1 \leq m_2 \leq \dots \leq m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}}$$

is the generalization of poly-logarithm, also known as multiple zeta values, which have been given much attention recently but until now, their precise structure remains a mystery. Note that, when $r = 1$, (6) immediately yields (4). Several identities on $E_n^{(k_1, k_2, \dots, k_r)}(x; a, b)$ have been obtained in [15] including the recurrence relations and certain explicit formula. However, this explicit formula is limited only to the case where $a = 1$ and $b = e$. In this present paper, more identities for $E_n^{(k)}(x; a, b, c)$ will be established and further generalization of $E_n^{(k_1, k_2, \dots, k_r)}(x; a, b)$ will be investigated.

2 Some Results on Generalized Poly-Euler Polynomials

The main objective of this section is to establish more identities for $E_n^{(k)}(x; a, b, c)$. First, let us consider an expression for $E_n^{(k)}(x; a, b, c)$ in terms of $E_i^{(k)}(a, b)$, $i = 0, 1, \dots, n$.

Theorem 2.1. *The generalized poly Euler polynomials satisfy the following relation*

$$E_n^{(k)}(x; a, b, c) = \sum_{i=0}^n \binom{n}{i} (\ln c)^{n-i} E_i^{(k)}(a, b) x^{n-i} \quad (7)$$

Proof. Using (4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} E_n^{(k)}(x; a, b, c) \frac{t^n}{n!} &= \frac{2\text{Li}_k(1 - (ab)^{-t})}{a^{-t} + b^t} c^{xt} = e^{xt \ln c} \sum_{n=0}^{\infty} E_n^{(k)}(a, b) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^n \frac{(xt \ln c)^{n-i}}{(n-i)!} E_i^{(k)}(a, b) \frac{t^i}{i!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} (\ln c)^{n-i} E_i^{(k)}(a, b) x^{n-i} \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. \square

The next identity gives a relation between $E_n^{(k)}(x; a, b, c)$ and $E_n^{(k)}(x)$.

Theorem 2.2. *The generalized poly Euler polynomials satisfy the following relation*

$$E_n^{(k)}(x; a, b, c) = (\ln a + \ln b)^n E_n^{(k)} \left(\frac{x \ln c + \ln a}{\ln a + \ln b} \right) \quad (8)$$

Proof. Using (4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} E_n^{(k)}(x; a, b, c) \frac{t^n}{n!} &= \frac{2\text{Li}_k(1 - (ab)^{-t})}{a^{-t}(1 + (ab)^t)} e^{xt \ln c} \\ &= 2e^{\frac{x \ln c + \ln a}{\ln ab} t \ln ab} \frac{\text{Li}_k(1 - e^{-t \ln ab})}{1 + e^{t \ln ab}} \\ &= \sum_{n=0}^{\infty} (\ln a + \ln b)^n E_n^{(k)} \left(\frac{x \ln c + \ln a}{\ln a + \ln b} \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. \square

Theorem 2.3. *The generalized poly-Euler polynomials satisfy the following relation*

$$\frac{d}{dx} E_{n+1}^{(k)}(x; a, b, c) = (n+1)(\ln c) E_n^{(k)}(x; a, b, c) \quad (9)$$

Proof. Using (4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{d}{dx} E_n^{(k)}(x; a, b, c) \frac{t^n}{n!} &= \frac{2t(\ln c)\text{Li}_k(1 - (ab)^{-t})}{(a^{-t} + b^t)} e^{xt \ln c} \\ \sum_{n=0}^{\infty} \frac{d}{dx} E_n^{(k)}(x; a, b, c) \frac{t^{n-1}}{n!} &= \sum_{n=0}^{\infty} (\ln c) E_n^{(k)}(x; a, b, c) \frac{t^n}{n!}. \end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d}{dx} E_{n+1}^{(k)}(x; a, b, c) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (\ln c) E_n^{(k)}(x; a, b, c) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. \square

The following corollary immediately follows from Theorem 2.3 by taking $c = e$. For brevity, let us denote $E_n^{(k)}(x; a, b, e)$ by $E_n^{(k)}(x; a, b)$.

Corollary 2.4. *The generalized poly-Euler polynomials are Appell polynomials in the sense that*

$$\frac{d}{dx} E_{n+1}^{(k)}(x; a, b) = (n+1) E_n^{(k)}(x; a, b) \quad (10)$$

Consequently, using the characterization of Appell polynomials [17, 18, 19], the following addition formula can easily be obtained.

Corollary 2.5. *The generalized poly-Euler polynomials satisfy the following addition formula*

$$E_n^{(k)}(x+y; a, b) = \sum_{i=0}^n \binom{n}{i} E_i^{(k)}(x; a, b) y^{n-i} \quad (11)$$

Taking $x = 0$ in formula (11) and using the fact that $E_n^{(k)}(0; a, b) = E_n^{(k)}(a, b)$, Corollary 2.5 gives formula (7) in Theorem 2.1 with $c = e$.

Furthermore, using the fact that $E_n^{(k)}(x; a, b)$ satisfies the relation

$$\sum_{n=0}^{\infty} E_n^{(k)}(x; a, b) \frac{t^n}{n!} = \frac{2\text{Li}_k(1 - (ab)^{-t})}{a^{-t} + b^t} e^{xt}, \quad (12)$$

one can easily derive the following identities by manipulating the right-hand side of (12) and by appropriate application of the Cauchy's rule for the product of power series

$$\begin{aligned} E_n^{(k)}(x; a, b) &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{l=m}^n \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \binom{n}{l} E_{n-l}^{(k)}(-m; a, b) (x)^{(m)} \\ &= \sum_{m=0}^{\infty} \sum_{l=m}^n \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \binom{n}{l} E_{n-l}^{(k)}(0; a, b) (x)_m \\ &= \sum_{m=0}^{\infty} \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{l}} \left\{ \begin{matrix} l+s \\ s \end{matrix} \right\} E_{n-m-l}^{(k)}(0; a, b) B_m^{(s)}(x) \\ &= \sum_{m=0}^{\infty} \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^s \binom{s}{j} (-\lambda)^{s-j} E_{n-m}^{(k)}(j; a, b) H_m^{(s)}(x; \lambda), \end{aligned}$$

where $(x)^{(n)} = x(x+1)\dots(x+n-1)$, $(x)_n = x(x-1)\dots(x-n+1)$,

$$\left(\frac{t}{e^t - 1} \right)^s e^{xt} = \sum_{n=0}^{\infty} B_n^{(s)}(x) \frac{t^n}{n!} \text{ and } \left(\frac{1-\lambda}{e^t - \lambda} \right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x; \lambda) \frac{t^n}{n!}.$$

Now, let us consider the following definition which contains certain symmetrized generalization of poly-Euler polynomials with parameters a , b and c .

Definition 2.6. For $m, n \geq 0$, we define

$$D_n^{(m)}(x, y; a, b, c) = \frac{1}{(\ln a + \ln b)^n} \sum_{k=0}^m \binom{m}{k} E_n^{(-k)}(x; a, b, c) \left(\frac{y \ln c + \ln a}{\ln a + \ln b} \right)^{m-k}. \quad (13)$$

The following theorem contains the double generating function for $D_n^{(m)}(x, y; a, b, c)$.

Theorem 2.7. For $n, m \geq 0$, we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_n^{(m)}(x, y; a, b, c) \frac{t^n u^m}{n! m!} = \frac{2e^{\left(\frac{y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{\left(\frac{x \ln c + \ln a}{\ln a + \ln b}\right)t} e^{t+u} (1 - e^{-t})}{(e^t + 1)(e^t + e^u - e^{t+u})}. \quad (14)$$

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_n^{(m)}(x, y; a, b, c) \frac{t^n u^m}{n! m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(\ln a + \ln b)^n} \sum_{k=0}^m E_n^{(-k)}(x; a, b, c) \left(\frac{y \ln c + \ln a}{\ln a + \ln b} \right)^{m-k} \frac{t^n}{n!} \frac{u^m}{k!(m-k)!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m \geq k} \frac{1}{(\ln a + \ln b)^n} E_n^{(-k)}(x; a, b, c) \left(\frac{y \ln c + \ln a}{\ln a + \ln b} \right)^{m-k} \frac{t^n}{n!} \frac{u^m}{k!(m-k)!}. \end{aligned}$$

Replacing $m - k$ with l and using Theorem 2.2, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_n^{(m)}(x, y; a, b, c) \frac{t^n u^m}{n! m!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(\ln a + \ln b)^n} E_n^{(-k)}(x; a, b, c) \left(\frac{y \ln c + \ln a}{\ln a + \ln b} \right)^l \frac{t^n}{n!} \frac{u^k}{k!} \frac{u^l}{l!} \\ &= e^{\left(\frac{y \ln c + \ln a}{\ln a + \ln b}\right)u} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} E_n^{(-k)} \left(\frac{x \ln c + \ln a}{\ln a + \ln b} \right) \frac{t^n}{n!} \frac{u^k}{k!} \\ &= e^{\left(\frac{y \ln c + \ln a}{\ln a + \ln b}\right)u} \sum_{k=0}^{\infty} \frac{2\text{Li}_k(1 - e^{-t})}{1 + e^{-t}} e^{\left(\frac{x \ln c + \ln a}{\ln a + \ln b}\right)t} \frac{u^k}{k!} \\ &= e^{\left(\frac{y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{\left(\frac{x \ln c + \ln a}{\ln a + \ln b}\right)t} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} E_n^{(-k)}(0) \frac{t^n}{n!} \frac{u^k}{k!}. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} E_n^{(-k)}(0) \frac{t^n}{n!} \frac{u^k}{k!} &= \sum_{k=0}^{\infty} \sum_{m>0} \frac{2(1 - e^{-t})^m m^k u^k}{1 + e^t k!} \\ &= \sum_{m>0} \frac{2(1 - e^{-t})^m}{1 + e^t} \sum_{k=0}^{\infty} \frac{(mu)^k}{k!} \\ &= \frac{2}{1 + e^t} \sum_{m>0} (1 - e^{-t})^m e^{mu} \\ &= \frac{2}{1 + e^t} \frac{(1 - e^{-t}) e^u}{1 - (1 - e^{-t}) e^u} = \frac{2e^{t+u} (1 - e^{-t})}{(e^t + 1)(e^t + e^u - e^{t+u})}. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_n^{(m)}(x, y; a, b, c) \frac{t^n u^m}{n! m!} = \frac{2e^{\left(\frac{y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{\left(\frac{x \ln c + \ln a}{\ln a + \ln b}\right)t} e^{t+u} (1 - e^{-t})}{(e^t + 1)(e^t + e^u - e^{t+u})}.$$

□

The following is an explicit formula for $D_n^{(m)}(x, y; a, b, c)$.

Theorem 2.8. *For $n, m \geq 0$, we have*

$$\begin{aligned} D_n^{(m)}(x, y; a, b, c) &= 2 \sum_{j=0}^{\infty} (j!)^2 \left(\sum_{l=0}^n \sum_{i=0}^{\infty} (-1)^i \frac{(\ln c^x a^{i+2} b^{i+1})^{n-l} - (\ln c^x a^{i+1} b^i)^{n-l}}{(\ln a + \ln b)^{n-l}} \binom{n}{l} \left\{ \begin{matrix} l \\ j \end{matrix} \right\} \right) \times \\ &\quad \times \left(\sum_{r=0}^m \left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b} \right)^{m-r} \binom{m}{r} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \right) \end{aligned}$$

Proof. Using Theorem 2.7,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_n^{(m)}(x, y; a, b, c) \frac{t^n u^m}{n! m!} &= \frac{2e^{\left(\frac{y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{\left(\frac{x \ln c + \ln a}{\ln a + \ln b}\right)t} e^{t+u} (1 - e^{-t})}{(e^t + 1)(1 - (e^t - 1)(e^u - 1))} \\ &= 2e^{\left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b}\right)u} e^{\left(\frac{x \ln c + 2 \ln a + \ln b}{\ln a + \ln b}\right)t} (1 - e^{-t}) \sum_{i=0}^{\infty} (-1)^i e^{it} \sum_{j=0}^{\infty} (e^t - 1)^j (e^u - 1)^j \\ &= 2e^{\left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b}\right)u} \sum_{i=0}^{\infty} (-1)^i e^{\left(\frac{x \ln c + (i+2) \ln a + (i+1) \ln b}{\ln a + \ln b}\right)t} \sum_{j=0}^{\infty} (e^t - 1)^j (e^u - 1)^j \\ &\quad - 2e^{\left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b}\right)u} \sum_{i=0}^{\infty} (-1)^i e^{\left(\frac{x \ln c + (i+1) \ln a + i \ln b}{\ln a + \ln b}\right)t} \sum_{j=0}^{\infty} (e^t - 1)^j (e^u - 1)^j. \end{aligned}$$

Applying the exponential generating function for Stirling numbers of the second kind [7]

$$\sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!},$$

we obtain

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_n^{(m)}(x, y; a, b, c) \frac{t^n u^m}{n! m!}$$

$$\begin{aligned}
&= 2 \sum_{j=0}^{\infty} \left(j! \sum_{i=0}^{\infty} (-1)^i \sum_{n=0}^{\infty} \frac{\left(\frac{x \ln c + (i+2) \ln a + (i+1) \ln b}{\ln a + \ln b} \right)^n t^n}{n!} \sum_{m=0}^{\infty} \left\{ m \right\}_j \frac{t^m}{m!} \right) \times \\
&\quad \times \left(j! \sum_{n=0}^{\infty} \frac{\left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b} \right)^n u^n}{n!} \sum_{m=0}^{\infty} \left\{ m \right\}_j \frac{u^m}{m!} \right) \\
&\quad - 2 \sum_{j=0}^{\infty} \left(j! \sum_{i=0}^{\infty} (-1)^i \sum_{n=0}^{\infty} \frac{\left(\frac{x \ln c + (i+1) \ln a + i \ln b}{\ln a + \ln b} \right)^n t^n}{n!} \sum_{m=0}^{\infty} \left\{ m \right\}_j \frac{t^m}{m!} \right) \times \\
&\quad \times \left(j! \sum_{n=0}^{\infty} \frac{\left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b} \right)^n u^n}{n!} \sum_{m=0}^{\infty} \left\{ m \right\}_j \frac{u^m}{m!} \right) \\
&= 2 \sum_{j=0}^{\infty} \left(j! \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^l (-1)^i \left(\frac{x \ln c + (i+2) \ln a + (i+1) \ln b}{\ln a + \ln b} \right)^{l-m} \binom{l}{m} \left\{ m \right\}_j \frac{t^l}{l!} \right) \times \\
&\quad \times \left(j! \sum_{p=0}^{\infty} \sum_{r=0}^p \left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b} \right)^{p-r} \binom{p}{r} \left\{ r \right\}_j \frac{u^p}{p!} \right) \\
&\quad - 2 \sum_{j=0}^{\infty} \left(j! \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^l (-1)^i \left(\frac{x \ln c + (i+1) \ln a + i \ln b}{\ln a + \ln b} \right)^{l-m} \binom{l}{m} \left\{ m \right\}_j \frac{t^l}{l!} \right) \times \\
&\quad \times \left(j! \sum_{p=0}^{\infty} \sum_{r=0}^p \left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b} \right)^{p-r} \binom{p}{r} \left\{ r \right\}_j \frac{u^p}{p!} \right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{n!} \frac{u^m}{m!} 2 \sum_{j=0}^{\infty} (j!)^2 \left(\sum_{l=0}^n \sum_{i=0}^{\infty} (-1)^i \left(\frac{x \ln c + (i+2) \ln a + (i+1) \ln b}{\ln a + \ln b} \right)^{n-l} \binom{n}{l} \left\{ l \right\}_j \right) \times \\
&\quad \times \left(\sum_{r=0}^m \left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b} \right)^{m-r} \binom{m}{r} \left\{ r \right\}_j \right) \\
&\quad - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{n!} \frac{u^m}{m!} 2 \sum_{j=0}^{\infty} (j!)^2 \left(\sum_{l=0}^n \sum_{i=0}^{\infty} (-1)^i \left(\frac{x \ln c + (i+1) \ln a + i \ln b}{\ln a + \ln b} \right)^{n-l} \binom{n}{l} \left\{ l \right\}_j \right) \times \\
&\quad \times \left(\sum_{r=0}^m \left(\frac{y \ln c + 2 \ln a + \ln b}{\ln a + \ln b} \right)^{m-r} \binom{m}{r} \left\{ r \right\}_j \right)
\end{aligned}$$

Comparing the coefficients yields the desired result. \square

3 Generalized Multi Poly-Euler Polynomials

Let us define a more general form of multi poly-Euler polynomials.

Definition 3.1. The generalized multi poly-Euler polynomials with parameters a , b and c are defined by

$$\frac{2\text{Li}_{(k_1, k_2, \dots, k_r)}(1 - (ab)^{-t})}{(a^{-t} + b^t)^r} c^{rxt} = \sum_{n=0}^{\infty} E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c) \frac{t^n}{n!}. \quad (15)$$

In particular,

$$\begin{aligned} E_n^{(k_1, k_2, \dots, k_r)}(x) &= E_n^{(k_1, k_2, \dots, k_r)}(x; 1, e, e) \\ E_n^{(k_1, k_2, \dots, k_r)}(a, b) &= E_n^{(k_1, k_2, \dots, k_r)}(0; a, b) \end{aligned}$$

The following theorem contains some identities for $E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c)$ which can be derived using the process in deriving the identities in Theorems 2.1 – 2.3.

Theorem 3.2. *The generalized multi poly-Euler polynomials satisfy the following relations*

$$\begin{aligned} E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c) &= \sum_{i=0}^n \binom{n}{i} (r \ln c)^{n-i} E_i^{(k_1, k_2, \dots, k_r)}(a, b) x^{n-i} \\ E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c) &= (\ln a + \ln b)^n E_n^{(k_1, k_2, \dots, k_r)} \left(\frac{rx \ln c + \ln a}{\ln a + \ln b} \right) \\ \frac{d}{dx} E_{n+1}^{(k_1, k_2, \dots, k_r)}(x; a, b, c) &= (n+1)(r \ln c) E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c) \end{aligned} \quad (16)$$

The characterization of Appell polynomials is supposed to be used to establish the addition formula for the generalized multi poly-Euler polynomials using (16). But the constant $r \ln c$ that appeared in the expression of (16) disqualifies $E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c)$ to be an Appell polynomial and to, consequently, satisfy any of the conditions of the said characterization. However, we can derive the addition formula using the same method in deriving the addition formula for $E_n^{(k_1, k_2, \dots, k_r)}(x; a, b)$ in [15]. The following theorem contains the addition formula for $E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c)$.

Theorem 3.3. *The generalized poly-Euler polynomials satisfy the following addition formula*

$$E_n^{(k_1, k_2, \dots, k_r)}(x + y; a, b, c) = \sum_{i=0}^n \binom{n}{i} (r \ln c)^{n-i} E_i^{(k_1, k_2, \dots, k_r)}(x; a, b, c) y^{n-i}.$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} E_n^{(k_1, k_2, \dots, k_r)}(x + y; a, b, c) \frac{t^n}{n!} &= \frac{2\text{Li}_{(k_1, k_2, \dots, k_r)}(1 - (ab)^{-t})}{(a^{-t} + b^t)^r} c^{(x+y)rt} \\ &= \frac{2\text{Li}_{(k_1, k_2, \dots, k_r)}(1 - (ab)^{-t})}{(a^{-t} + b^t)^r} c^{xrt} c^{yrt} \\ &= \left(\sum_{n=0}^{\infty} E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (yr \ln c)^n \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} (yr \ln c)^{n-i} E_i^{(k_1, k_2, \dots, k_r)}(x; a, b, c) \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired result. \square

The next theorem contains an explicit formula for $E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c)$.

Theorem 3.4. *The generalized multi poly-Euler polynomials have the following explicit formula*

$$E_n^{(k_1, k_2, \dots, k_r)}(x; a, b, c) = \sum_{i=0}^n \sum_{\substack{0 \leq m_1 \leq m_2 \leq \dots \leq m_r \\ c_1 + c_2 + \dots = r}} \sum_{j=0}^{m_r} \frac{2(rx \ln c - j \ln ab)^{n-i} r! (-1)^{j+s} (s \ln ab + r \ln a)^i \binom{m_r}{j} \binom{n}{i}}{(c_1! c_2! \dots) (m_1^{k_1} m_2^{k_2} \dots m_r^{k_r})}, \quad (17)$$

where $s = c_1 + 2c_2 + \dots$

Proof. From Definition 3.1, we have

$$\begin{aligned} \text{Li}_{(k_1, k_2, \dots, k_r)}(1 - (ab)^{-t}) c^{rxt} &= \sum_{0 \leq m_1 \leq m_2 \leq \dots \leq m_r} \frac{(1 - (ab)^{-t})^{m_r}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}} e^{rxt \ln c} \\ &= \sum_{0 \leq m_1 \leq m_2 \leq \dots \leq m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}} \sum_{j=0}^{m_r} (-1)^j \binom{m_r}{j} \sum_{n=0}^{\infty} (rx \ln c - j \ln ab)^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{0 \leq m_1 \leq m_2 \leq \dots \leq m_r} \sum_{j=0}^{m_r} \frac{(-1)^j (rx \ln c - j \ln ab)^n \binom{m_r}{j}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}} \right) \frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left(\frac{1}{a^{-t} + b^t} \right)^r &= a^{rt} \left(\frac{1}{1 + (ab)^t} \right)^r = a^{rt} \left(\sum_{n \geq 0} (-1)^n (ab)^{nt} \right)^r \\ &= \sum_{c_1 + c_2 + \dots = r} \frac{r! (-1)^{c_1 + 2c_2 + \dots}}{c_1! c_2! \dots} e^{t[r \ln a + (c_1 + 2c_2 + \dots) \ln ab]} \\ &= \sum_{c_1 + c_2 + \dots = r} \frac{r! (-1)^{c_1 + 2c_2 + \dots}}{c_1! c_2! \dots} \sum_{n=0}^{\infty} (r \ln a + (c_1 + 2c_2 + \dots) \ln ab)^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{c_1 + c_2 + \dots = r} \frac{r! (-1)^{c_1 + 2c_2 + \dots} (r \ln a + (c_1 + 2c_2 + \dots) \ln ab)^n}{c_1! c_2! \dots} \right) \frac{t^n}{n!}. \end{aligned}$$

Hence,

$$\frac{2\text{Li}_{(k_1, k_2, \dots, k_r)}(1 - (ab)^{-t})}{(a^{-t} + b^t)^r} c^{rxt} = 2\text{Li}_{(k_1, k_2, \dots, k_r)}(1 - (ab)^{-t}) e^{rxt \ln c} a^{rt} \left(\frac{1}{1 + (ab)^t} \right)^r$$

$$\begin{aligned}
&= 2 \sum_{n=0}^{\infty} \sum_{i=0}^n \left(\sum_{0 \leq m_1 \leq m_2 \leq \dots \leq m_r} \sum_{j=0}^{m_r} \frac{(-1)^j (rx \ln c - j \ln ab)^{n-i} \binom{m_r}{j}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}} \right) \frac{t^{n-i}}{(n-i)!} \times \\
&\quad \times \left(\sum_{c_1+c_2+\dots=r} \frac{r!(-1)^{c_1+2c_2+\dots} (r \ln a + (c_1+2c_2+\dots) \ln ab)^i}{c_1! c_2! \dots} \right) \frac{t^i}{i!} \\
&= 2 \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{\substack{0 \leq m_1 \leq m_2 \leq \dots \leq m_r \\ c_1+c_2+\dots=r}} \sum_{j=0}^{m_r} \frac{H(r, i, j, n, a, b)}{(c_1! c_2! \dots) (m_1^{k_1} m_2^{k_2} \dots m_r^{k_r})} \frac{t^n}{n!}
\end{aligned}$$

where

$$H(r, i, j, n, a, b) = (rx \ln c - j \ln ab)^{n-i} r! (-1)^{j+c_1+2c_2+\dots} (r \ln a + (c_1+2c_2+\dots) \ln ab)^i \binom{m_r}{j} \binom{n}{i}.$$

By comparing the coefficient of $t^n/n!$, we obtain the desired explicit formula. \square

Definition 3.5. For $m, n \geq 0$, we define

$$\mathcal{D}_n^{(m)}(x, y; a, b, c) = \sum_{k_1+k_2+\dots+k_r=m} \binom{m}{k_1, k_2, \dots, k_r} \frac{E_n^{(-k_1, -k_2, \dots, -k_{r-1})}(x; a, b, c)}{(\ln a + \ln b)^n} \left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b} \right)^{k_r}. \quad (18)$$

The following theorem contains the double generating function for $\mathcal{D}_n^{(m)}(x, y; a, b, c)$.

Theorem 3.6. For $n, m \geq 0$, we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{D}_n^{(m)}(x, y; a, b, c) \frac{t^n}{n!} \frac{u^m}{m!} = \frac{2e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t} e^{\binom{r}{2}u + (r-1)t} (1 - e^{-t})^{r-1}}{(1 + e^t)^{r-1} \prod_{i=1}^{r-1} (e^t + e^{iu} - e^{t+iu})}. \quad (19)$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{D}_n^{(m)}(x, y; a, b, c) \frac{t^n}{n!} \frac{u^m}{m!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k_1+k_2+\dots+k_r=m} \frac{E_n^{(-k_1, -k_2, \dots, -k_{r-1})}(x; a, b, c)}{(\ln a + \ln b)^n} \left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b} \right)^{k_r} \frac{t^n}{n!} \frac{u^m}{k_1! k_2! \dots k_r!} \\
&= \sum_{n=0}^{\infty} \sum_{k_1+k_2+\dots+k_r \geq 0} \frac{E_n^{(-k_1, -k_2, \dots, -k_{r-1})}(x; a, b, c)}{(\ln a + \ln b)^n} \left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b} \right)^{k_r} \frac{t^n}{n!} \frac{u^{k_1+k_2+\dots+k_r}}{k_1! k_2! \dots k_r!} \\
&= \sum_{n=0}^{\infty} \sum_{k_1+k_2+\dots+k_{r-1} \geq 0} \frac{E_n^{(-k_1, -k_2, \dots, -k_{r-1})}(x; a, b, c)}{(\ln a + \ln b)^n} \sum_{k_r \geq 0} \left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b} \right)^{k_r} \frac{u^{k_r}}{k_r!} \frac{t^n}{n!} \frac{u^{k_1+k_2+\dots+k_{r-1}}}{k_1! k_2! \dots k_{r-1}!} \\
&= e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} \sum_{n=0}^{\infty} \sum_{k_1+k_2+\dots+k_{r-1} \geq 0} \frac{E_n^{(-k_1, -k_2, \dots, -k_{r-1})}(x; a, b, c)}{(\ln a + \ln b)^n} \frac{t^n}{n!} \frac{u^{k_1+k_2+\dots+k_{r-1}}}{k_1! k_2! \dots k_{r-1}!}
\end{aligned}$$

Using Theorem 3.2, we obtain

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{D}_n^{(m)}(x, y; a, b, c) \frac{t^n}{n!} \frac{u^m}{m!} \\
&= e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} \sum_{k_1+k_2+\dots+k_{r-1} \geq 0} \sum_{n=0}^{\infty} E_n^{(-k_1, -k_2, \dots, -k_{r-1})} \left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b} \right) \frac{t^n}{n!} \frac{u^{k_1+k_2+\dots+k_{r-1}}}{k_1!k_2! \dots k_{r-1}!} \\
&= e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t} \sum_{k_1+k_2+\dots+k_{r-1} \geq 0} \frac{2\text{Li}_{(-k_1, -k_2, \dots, -k_{r-1})}(1 - e^{-t})}{(1 + e^t)^{r-1}} \frac{u^{k_1+k_2+\dots+k_{r-1}}}{k_1!k_2! \dots k_{r-1}!} \\
&= \frac{2e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t}}{(1 + e^t)^{r-1}} \sum_{0 < m_1 < m_2 < \dots < m_{r-1}} (1 - e^{-t})^{m_{r-1}} \mathcal{S}(u, m_1, m_2, \dots, m_{r-1})
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}(u, m_1, m_2, \dots, m_{r-1}) &= \sum_{k_1+k_2+\dots+k_{r-1} \geq 0} \frac{(um_1)^{k_1} (um_2)^{k_2} \dots (um_{r-1})^{k_{r-1}}}{k_1!k_2! \dots k_{r-1}!} \\
&= \sum_{\hat{m} \geq 0} \frac{1}{\hat{m}!} \sum_{k_1+k_2+\dots+k_{r-1} = \hat{m}} \binom{\hat{m}}{k_1, k_2, \dots, k_{r-1}} (um_1)^{k_1} (um_2)^{k_2} \dots (um_{r-1})^{k_{r-1}} \\
&= \sum_{\hat{m} \geq 0} \frac{(um_1 + um_2 + \dots + um_{r-1})^{\hat{m}}}{\hat{m}!} \\
&= e^{u(m_1+m_2+\dots+m_{r-1})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{D}_n^{(m)}(x, y; a, b, c) \frac{t^n}{n!} \frac{u^m}{m!} \\
&= \frac{2e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t}}{(1 + e^t)^{r-1}} \sum_{0 < m_1 < m_2 < \dots < m_{r-1}} (1 - e^{-t})^{m_{r-1}} e^{u(m_1+m_2+\dots+m_{r-1})} \\
&= \frac{2e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t}}{(1 + e^t)^{r-1}} \frac{e^u(1 - e^{-t})}{1 - e^u(1 - e^{-t})} \frac{e^{2u}(1 - e^{-t})}{1 - e^{2u}(1 - e^{-t})} \dots \frac{e^{(r-1)u}(1 - e^{-t})}{1 - e^{(r-1)u}(1 - e^{-t})} \\
&= \frac{2e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t} e^{\binom{r}{2}u(1 - e^{-t})}}{(1 + e^t)^{r-1} \prod_{i=1}^{r-1} (1 - e^{iu}(1 - e^{-t}))} \\
&= \frac{2e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t} e^{\binom{r}{2}u + (r-1)t} (1 - e^{-t})^{r-1}}{(1 + e^t)^{r-1} \prod_{i=1}^{r-1} (e^t + e^{iu} - e^{t+iu})}.
\end{aligned}$$

□

Note that equation (14) can easily be deduced from equation (19) by taking $r = 1$. It is then interesting to establish an explicit formula for $\mathcal{D}_n^{(m)}(x, y; a, b, c)$ parallel to Theorem 2.8. To do this, let us consider first the following expression from the right-hand side of equation (19). That is,

$$\begin{aligned}
\frac{1}{(1+e^t)^{r-1} \prod_{i=1}^{r-1} (e^t + e^{iu} - e^{t+iu})} &= \left(\sum_{n=0}^{\infty} (-e)^{nt} \right)^{r-1} \prod_{i=1}^{r-1} \frac{1}{(1 - (e^t - 1)(e^{iu} - 1))} \\
&= \left(\sum_{n=0}^{\infty} (-e)^{nt} \right)^{r-1} \prod_{i=1}^{r-1} \sum_{j=0}^{\infty} (e^t - 1)^j (e^{iu} - 1)^j \\
&= \left(\sum_{n=0}^{\infty} (-e)^{nt} \right)^{r-1} \prod_{i=1}^{r-1} \sum_{c_i \geq 0} (e^t - 1)^{c_i} (e^{iu} - 1)^{c_i} \\
\left(\sum_{n=0}^{\infty} (-e)^{nt} \right)^{r-1} &= \sum_{q=0}^{\infty} \sum_{k_{r-1}=0}^n \sum_{k_{r-2}=0}^{n-k_{r-1}} \cdots \sum_{k_1=0}^{n-k_{r-1}-\dots-k_2} (-1)^q e^{qt} \\
&= \sum_{q=0}^{\infty} (-1)^q \frac{\prod_{j=0}^{q-2} (q+1+j)}{(q-1)!} e^{qt}
\end{aligned}$$

$$\begin{aligned}
\prod_{i=1}^{r-1} \sum_{c_i \geq 0} (e^t - 1)^{c_i} (e^{iu} - 1)^{c_i} &= \sum_{j=0}^{\infty} \sum_{c_1+c_2+\dots+c_{r-1}=j} (e^t - 1)^j \prod_{i=1}^{r-1} (e^{iu} - 1)^{c_i} \\
&= \sum_{j=0}^{\infty} \sum_{c_1+c_2+\dots+c_{r-1}=j} j! \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \frac{t^n}{n!} \prod_{i=1}^{r-1} c_i! \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ c_i \end{matrix} \right\} \frac{u^n}{n!}
\end{aligned}$$

$$\begin{aligned}
\prod_{i=1}^{r-1} c_i! \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ c_i \end{matrix} \right\} \frac{u^n}{n!} &= \sum_{m=0}^{\infty} \sum_{d_{r-2}=0}^m \sum_{d_{r-3}=0}^{m-d_{r-2}} \cdots \sum_{d_1=0}^{m-d_{r-2}-\dots-d_2} c_1! \left\{ \begin{matrix} m-d_{r-2}-\dots-d_1 \\ c_1 \end{matrix} \right\} \times \\
&\times \prod_{i=1}^{r-2} \left(\begin{matrix} m-d_{r-2}-\dots-d_{i+1} \\ d_i \end{matrix} \right) \left\{ \begin{matrix} d_i \\ c_{i+1} \end{matrix} \right\} c_{i+1}! (i+1)^{d_i} \frac{u^m}{m!}
\end{aligned}$$

$$e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t} (1-e^{-t})^{r-1} \left(\sum_{n=0}^{\infty} (-e)^{nt} \right)^{r-1} \left(j! \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \frac{t^n}{n!} \right)$$

$$\begin{aligned}
&= e^{(r-1)\left(\frac{(r-1)x \ln c + \ln a}{\ln a + \ln b}\right)t} \left(\sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} e^{-kt} \right) \left(\sum_{q=0}^{\infty} (-1)^q \frac{\prod_{j=0}^{q-2} (q+1+j)}{(q-1)!} e^{qt} \right) \left(j! \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \frac{t^n}{n!} \right) \\
&= \left(\sum_{q=0}^{\infty} \sum_{k=0}^{r-1} e^{\left(\frac{(r-1)^2 x \ln c + (q-k) \ln b + (q-k+r-1) \ln a}{\ln a + \ln b} \right)} \frac{(-1)^{k+q} \binom{r-1}{k} \prod_{j=0}^{q-2} (q+1+j)}{(q-1)!} \right) \left(j! \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \frac{t^n}{n!} \right) \\
&= \left(\sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \sum_{k=0}^{r-1} \left(\frac{(r-1)^2 x \ln c + (q-k) \ln b + (q-k+r-1) \ln a}{\ln a + \ln b} \right)^n \right. \\
&\quad \left. \frac{(-1)^{k+q} \binom{r-1}{k} \prod_{j=0}^{q-2} (q+1+j)}{(q-1)!} \frac{t^n}{n!} \right) \left(j! \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{p=0}^n \sum_{q=0}^{\infty} \sum_{k=0}^{r-1} \binom{n}{p} \left(\frac{(r-1)^2 x \ln c + (q-k) \ln b + (q-k+r-1) \ln a}{\ln a + \ln b} \right)^{n-p} \right. \\
&\quad \left. \frac{(-1)^{k+q} \binom{r-1}{k} \prod_{j=0}^{q-2} (q+1+j)}{(q-1)!} j! \left\{ \begin{matrix} p \\ j \end{matrix} \right\} \right) \frac{t^n}{n!} \\
&e^{\left(\frac{(r-1)y \ln c + \ln a}{\ln a + \ln b} \right)u} e^{\binom{r}{2}u} \left(\prod_{i=1}^{r-1} c_i! \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ c_i \end{matrix} \right\} \frac{u^n}{n!} \right) \\
&= \left(\sum_{m=0}^{\infty} \left(\frac{(r-1)y \ln c + \binom{r}{2} \ln b + \left\{ \binom{r}{2} + 1 \right\} \ln a}{\ln a + \ln b} \right)^m \frac{u^m}{m!} \right) \left(\sum_{m=0}^{\infty} \sum_{d_{r-2}=0}^m \sum_{d_{r-3}=0}^{m-d_{r-2}} \dots \right. \\
&\quad \left. \sum_{d_1=0}^{m-d_{r-2}-\dots-d_2} c_1! \left\{ \begin{matrix} m-d_{r-2}-\dots-d_1 \\ c_1 \end{matrix} \right\} \prod_{i=1}^{r-2} \binom{m-d_{r-2}-\dots-d_{i+1}}{d_i} \left\{ \begin{matrix} d_i \\ c_{i+1} \end{matrix} \right\} c_{i+1}! (i+1)^{d_i} \frac{u^m}{m!} \right) \\
&= \sum_{m=0}^{\infty} \left\{ \sum_{l=0}^m \left(\frac{(r-1)y \ln c + \binom{r}{2} \ln b + \left\{ \binom{r}{2} + 1 \right\} \ln a}{\ln a + \ln b} \right)^{m-l} \sum_{d_{r-2}=0}^l \sum_{d_{r-3}=0}^{l-d_{r-2}} \dots \right. \\
&\quad \left. \sum_{d_1=0}^{l-d_{r-2}-\dots-d_2} c_1! \left\{ \begin{matrix} l-d_{r-2}-\dots-d_1 \\ c_1 \end{matrix} \right\} \prod_{i=1}^{r-2} \binom{l-d_{r-2}-\dots-d_{i+1}}{d_i} \left\{ \begin{matrix} d_i \\ c_{i+1} \end{matrix} \right\} c_{i+1}! (i+1)^{d_i} \right\} \frac{u^m}{m!} \\
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{D}_n^{(m)}(x, y; a, b, c) \frac{t^n}{n!} \frac{u^m}{m!}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{j=0}^{\infty} \sum_{c_1+c_2+\dots+c_{r-1}=j} \sum_{l=0}^m \left(\frac{(r-1)y \ln c + \binom{r}{2} \ln b + \left\{ \binom{r}{2} + 1 \right\} \ln a}{\ln a + \ln b} \right)^{m-l} \right. \\
&\quad \sum_{d_{r-2}=0}^l \sum_{d_{r-3}=0}^{l-d_{r-2}} \dots \sum_{d_1=0}^{l-d_{r-2}-\dots-d_2} c_1! \left\{ \begin{matrix} l-d_{r-2}-\dots-d_1 \\ c_1 \end{matrix} \right\} \prod_{i=1}^{r-2} \binom{l-d_{r-2}-\dots-d_{i+1}}{d_i} \times \\
&\quad \times \left\{ \begin{matrix} d_i \\ c_{i+1} \end{matrix} \right\} c_{i+1}! (i+1)^{d_i} \sum_{p=0}^n \sum_{q=0}^{\infty} \sum_{k=0}^{r-1} \binom{n}{p} \left(\frac{(r-1)^2 x \ln c + (q-k) \ln b + (q-k+r-1) \ln a}{\ln a + \ln b} \right)^{n-p} \\
&\quad \times \frac{(-1)^{k+q} \binom{r-1}{k} \prod_{j=0}^{q-2} (q+1+j)}{(q-1)!} j! \left\{ \begin{matrix} p \\ j \end{matrix} \right\} \left. \right\} \frac{t^n}{n!} \frac{u^m}{m!}
\end{aligned}$$

Comparing coefficients, we obtain the following theorem.

Theorem 3.7. *For $n, m \geq 0$, we have*

$$\begin{aligned}
\mathcal{D}_n^{(m)}(x, y; a, b, c) &= \sum_{j=0}^{\infty} \sum_{c_1+c_2+\dots+c_{r-1}=j} \sum_{l=0}^m \left(\frac{(r-1)y \ln c + \binom{r}{2} \ln b + \left\{ \binom{r}{2} + 1 \right\} \ln a}{\ln a + \ln b} \right)^{m-l} \times \\
&\times \sum_{d_{r-2}=0}^l \sum_{d_{r-3}=0}^{l-d_{r-2}} \dots \sum_{d_1=0}^{l-d_{r-2}-\dots-d_2} c_1! \left\{ \begin{matrix} l-d_{r-2}-\dots-d_1 \\ c_1 \end{matrix} \right\} \prod_{i=1}^{r-2} \binom{l-d_{r-2}-\dots-d_{i+1}}{d_i} \times \\
&\times \left\{ \begin{matrix} d_i \\ c_{i+1} \end{matrix} \right\} c_{i+1}! (i+1)^{d_i} \sum_{p=0}^n \sum_{q=0}^{\infty} \sum_{k=0}^{r-1} \binom{n}{p} \left(\frac{(r-1)^2 x \ln c + (q-k) \ln b + (q-k+r-1) \ln a}{\ln a + \ln b} \right)^{n-p} \times \\
&\times \frac{(-1)^{k+q} \binom{r-1}{k} \prod_{j=0}^{q-2} (q+1+j)}{(q-1)!} j! \left\{ \begin{matrix} p \\ j \end{matrix} \right\}.
\end{aligned}$$

References

- [1] Serkan Araci, Mehmet Acikgoz and Erdogan Sen, On the extended Kims p -adic q -deformed fermionic integrals in the p -adic integer ring, *J. of Number Theory*, **133**(10) (2013), 33483361.
- [2] A. Bayad and Y. Hamahata, Arakawa-Kaneko L -functions and generalized poly-Bernoulli polynomials, *J. Number Theory*, **131** (2011), 10201036.
- [3] A. Bayad and Y. Hamahata, Multiple polylogarithms and multi-poly-Bernoulli polynomials, *Funct. Approx. Comment. Math.*, **46**(1) (2012), 45-61.
- [4] B. Beñyi, Advances in Bijective Combinatorics, *Ph.D. Thesis*, 2014.
- [5] C. Brewbaker, A Combinatorial Interpretation of the Poly-Bernoulli Numbers and Two Fermat Analogues, *Integers*, **8** (2008), #A02

- [6] B.Candelpergher and M. A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *The Ramanujan Journal*, **27**(3) (2012), 305-328.
- [7] Comtet, L., Advanced Combinatorics, *D. Reidel Publishing Company*, 1974.
- [8] M-A. Coppo and B. Candelpergher, The Arakawa-Kaneko Zeta Function, *The Ramanujan Journal*, **22** (2010), 153-162.
- [9] Y. Hamahata, Poly-Euler Polynomials and Arakawa-Kaneko Type Zeta Functions, *Funct. Approx. Comment. Math.*, **51**(1) (2014), 7-22.
- [10] L. Jang, T. Kim, and H. K. Pak, A note on q -Euler and Genocchi numbers, *Proc. Japan Acad. Ser. A Math. Sci.*, **77**(8) (2001), 139-141.
- [11] M. Kaneko, Poly-Bernoulli numbers, *J. Theorie de Nombres*, **9** (1997), 221–228.
- [12] T. Kim, q -Generalized Euler numbers and polynomials, **13**(3) (2006), 293-298.
- [13] H. Jolany, R. E. Alikelaye and S. S. Mohamad, Some Results on the Generalization of Bernoulli, Euler and Genocchi Polynomials, *Acta Universitatis Apulensis*, **27** (2011), pp. 299–306.
- [14] H. Jolany, M. Aliabadi, R. B. Corcino and M.R.Darafsheh, A Note on Multi Poly-Euler Numbers and Bernoulli Polynomials, *General Mathematics*, **20**(2-3) (2012), 122-134
- [15] H. Jolany, M.R. Darafsheh, R.E. Alikelaye, Generalizations of Poly-Bernoulli Numbers and Polynomials, *Int. J. Math. Comb.*, **2** (2010), 7–14.
- [16] Y. Ohno and Y. Sasaki, On the parity of poly-Euler numbers, *RIMS Kokyuroku Bessatsu*, **B32** (2012), 271278.
- [17] D. W. Lee, On Multiple Appell Polynomials, *Proceedings of the American Mathematical Society*, **139**(6) (2011), 2133-2141.
- [18] J. Shohat, The Relation of the Classical Orthogonal Polynomials to the Polynomials of Appell, *Amer. J. Math.*, **58** (1936), 453-464
- [19] L. Toscano, Polinomi Ortogonali o Reciproci di Ortogonali Nella classe di Appell, *Le Matematica*, **11** (1956), 168-174.

Hassan Jolany

Université des Sciences et Technologies de Lille

UFR de Mathématiques

Laboratoire Paul Painlevé

CNRS-UMR 8524 59655 Villeneuve d’Ascq Cedex/France

e-mail: hassan.jolany@math.univ-lille1.fr

Roberto B. Corcino

Mathematics and ICT Department

Cebu Normal University

Osmena Blvd., Cebu City

Philippines 6000

e-mail: rcorcino@yahoo.com